STABILITY OF CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

V. M. Kornev and N. D. Mel'nikova

UDC 539.3

By methods of the theory of perturbations [1-4], upper critical loads have been obtained for nonideal cylindrical shells under transverse and hydrostatic loading. Problems of approximation are studied in particular; when determining the total and ordinal numbers of degrees of freedom, the information about the density of the spectrum of the corresponding linear stability problems is used [5-7]. The band of scatter of the upper critical load is obtained. A numerical experiment allowed the probability characteristics of the process of stability loss to be calculated.

We consider a nonlinear system of equations of the theory of shells which relative to the stress function F and normal deflection w describes the buckling of ideal thin-walled cylindrical shells under transverse or hydrostatic loading (see, for example, [4]). According to the perturbation theory, the functions w and F and the loading parameter λ are expanded in the asymptotic series

$$\varkappa = \lambda/\lambda_1 = 1 + a\varepsilon + b\varepsilon^2 + \ldots, \quad \varepsilon \ll 1, \tag{1}$$

where ε is a small parameter characterizing the amplitude of the buckled state; λ_1 is the first eigenvalue of the linear problem; a and b are coefficients $- a \equiv 0$, while the values of the coefficient b have been obtained in [4].

Knowing the coefficient b we can calculate the upper critical load λ_+ of the ideal construction [4] in the case of the single-term approximation

$$\left(1 - \frac{\lambda_+}{\lambda_1}\right)^{3/2} = \frac{3}{2}^{3/2} |f_1| \sqrt{-b}\, \frac{\lambda_+}{\lambda_1}. \tag{2}$$

Here $|f_1|$ is the dimensionless amplitude of a fault when the fault coincides with the first mode of stability loss.

The present paper supplements the results of the works [2-4] on stability of nonideal cylindrical shells: 1) We study the spectrum in stability problems for different methods of fixing the ends of cylinders [5, 8, 9]; 2) we determine the total and ordinal numbers of equivalent modes of stability loss for nonhomogeneous linear problems [6, 7]; 3) we have chosen the corresponding single-term approximation for a system with several degrees of freedom.

The problems of approximation of nonideal systems with distributed parameters [6, 7] are directly connected with the character of the spectrum in the vicinity of the least eigenvalue λ_1. In this vicinity the expression for the critical loads q(n, 1) is simplified:

$$\begin{gathered} q(n, 1) = \frac{n^2}{12(1-\nu^2)} \frac{t}{R} + \frac{\pi^4 R^4}{L^4} \frac{R}{t}, \quad \left(\frac{\pi R}{n_* L}\right)^2 \ll 1, \\ \min q(n, 1) = q(n_*, 1), \quad n_*^2 \approx \pi \left[6(1-\nu^2)^{1/2}\right]^{1/2} (R/L)(R/t)^{1/2}, \end{gathered} \tag{3}$$

where L, R, and t are the length, radius, and thickness of the cylindrical shell; ν is Poisson's ratio. It is obvious that the number of waves along the circumferential coordinate is $n_* \gg 1$ for sufficiently thin-walled shells of medium length ($L/R = O(1)$). Consequently, the adjacent modes with the numbers $n_1, n_1 + 1, \ldots, n_*, \ldots, n_2$ are matched by the critical parameters $q(n_1, 1), \ldots, q(n_*, 1), \ldots, q(n_2, 1)$, which differ little if $n_0 \ll n_*$, $n_0 = \max(n_* - n_1, n_2 - n_*)$. We introduce the notation

$$n = n_*(1 \pm \mu), \quad \mu = n_0/n_* \ll 1, \quad n_0 = \max(n_* - n_1, n_2 - n_*). \tag{4}$$

If the relation (4) is inserted into the expression (3) and the second-degree small terms are neglected, then we have $q(n, 1) = (1 + 1.5\mu^2)q(n_*, 1)$. Approximately the number of degrees of freedom [6, 7] which are equivalent in asymptotic expansions for nonhomogeneous linear buckling problems, when all Fourier coefficients of the

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki Tekhnicheskoi Fiziki, No. 2, pp. 145-150, March-April, 1978. Original article submitted December 29, 1976.

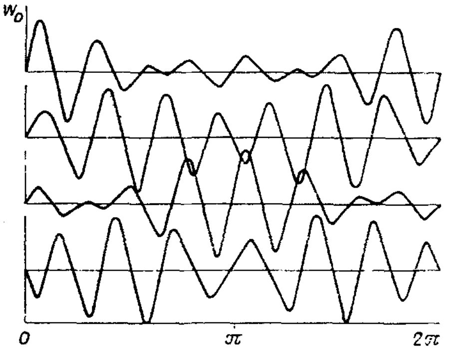

Fig. 1

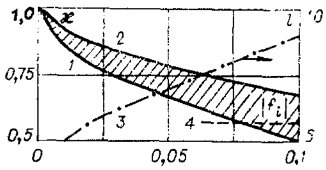

Fig. 2

right sides are of the same order of smallness, is calculated according to the expressions

$$l = 1 + 2\mu n_*, \quad \mu = \left(\frac{1-\varkappa_+}{1.5}\frac{1-\beta}{\beta}\right)^{1/2}, \quad \varkappa_+ = \frac{\lambda_+}{\lambda_1}, \tag{5}$$

where β is the coefficient characterizing the ratio of the amplitude of the mode ($n_* \pm n_0$, 1) to the amplitude of the mode (n_*, 1) in the nonhomogeneous buckling problem. In the calculations, this coefficient β is taken equal to 0.9.

When obtaining the expression (5) it was assumed that the faults corresponding to the modes of buckling with the numbers

$$n_1,\ n_1+1, \ldots, n_2 \text{ for } m = 1;\ n_1 \le n_* \le n_2,\ l = 1 + n_2 - n_1, \tag{6}$$

have the same order of smallness. Since $n_1 = O(n_*)$, $n_2 = O(n_*)$, $n_* = O[(R/t)^{1/4}]$, the given assumption is natural: The coefficients of the Fourier series with sufficiently high numbers from n_1 to n_2 are commensurable. We note that the last assumption in other problems will not be natural (see, for example, [6], where a nonhomogeneous problem for the Klein–Gordon equation is studied).

Thus, we consider shells whose initial deflection is given in the form of the trigonometric polynomial

$$W_0 = t\sin\frac{\pi x}{L}\sum_{i=n_1}^{n_2} f_i \sin\frac{i(y+y_{i0})}{R}. \tag{7}$$

In contrast to [2-4], we consider not any single fault coinciding with the first mode of stability loss, but an entire class of faults W_0 corresponding to equivalent modes of stability loss (6); axisymmetrical faults are excluded from the consideration. It is obvious that functions from the class (7) have the character of beating with respect to the circumferential coordinate ($n_* \gg 1$).

We next consider a class of faults (7) in which $|f_i| = |f_{n_*}|$, $y_{i0} \equiv 0$, i.e., one fault differs from another by signs of the coefficients of the segment of a Fourier series (Fig. 1). Each of the functions W_0 in the vicinity of the maximum value is well described by the function

$$W = tf_* \sin(\pi x/L)\sin n_*(y + y_*)R^{-1},\ f_* = \pm \max W_0. \tag{8}$$

Indeed, transferring the origin of reference to the point where the function W_0 is maximum, we obtain

$$W_0 = t|f_{n_*}|\sin\frac{\pi x}{L}\sum_{j=n_1-n_*}^{n_2-n_*} c_i \sin\left[\left(\frac{n_* z}{R} + \varphi_i\right) + j\frac{z}{R}\right],\quad |c_i| = 1, \tag{9}$$

where φ_i is the initial phase of the sinusoidal quantity. If we transform the sine under the sum sign according to the expression of sum of angles, then for z, for which $|jz/R| \ll 1$, the transformed expression (9) is considerably simplified. Bearing in mind that a linear combination of several sinusoidal quantities $c_i \sin(n_* z/R + \varphi_i)$ with the same frequency is a sinusoidal quantity of the same frequency, we have

$$W_0 = t\sin\frac{\pi x}{L}\left[f_*\sin\left(\frac{n_* z}{R} + \varphi\right) + A\right],$$
$$|A| \le |f_{n_*}|\sum_{j=n_1-n_*}^{n_2-n_*}|j|\frac{z}{R} < 2n_0^2\frac{z}{R}|f_{n_*}|. \tag{10}$$

TABLE 1

$\|f_i\|$	0,025			0,05			0,1		
L/R	l	$\varkappa_1$	$\varkappa_2$	l	$\varkappa_1$	$\varkappa_2$	l	$\varkappa_1$	$\varkappa_2$
2	2	0,937	0,937	2	0,903	0,903	3	0,840	0,812
1	3	0,892	0,872	4	0,819	0,772	5	0,733	0,636
0,5	5	0,847	0,778	6	0,755	0,650	7	0,619	0,507

From the relation (10) we have (8).

Going over to the calculation of the upper critical load λ_+, for systems with many degrees of freedom (6) we obtain an estimate for this parameter λ_+ from below. In each concrete case the initial fault W_0 has already been estimated by (8). Therefore, we can use the expression (2) for the calculations, but instead of the dimensionless amplitude $|f_1|$, corresponding to the first mode of stability loss, we must insert the dimensionless amplitude of the initial fault W_0, i.e., $|f_*|$. The approach being proposed is founded on the experimental work [10, 11]. In [11] it is mentioned that ". . . buckling, generally speaking – a very localized phenomenon – does not depend on the adjacent regions of the shell which may or may not be enveloped by the subsequent stages of buckling."

In Table 1, for a shell with $R/t = 300$, we have presented the results of calculations to determine the number of degrees of freedom l, the dimensionless parameter $\varkappa_-$ ($\varkappa_+ = \lambda_+/\lambda_1$) dependent on the dimensionless amplitudes $|f_i|$ and the relative length of the shell L/R [see (2), (5), (7)] (in the table and subsequently the sign on the parameter $\varkappa_+$ is omitted, $\varkappa_1 = \max \varkappa_+$, $\varkappa_2 = \min \varkappa_+$). From the calculations thus presented we see that the character of the spectrum in a substantial manner influences the number of degrees of freedom l. With the growth of the amplitudes of the initial faults $|f_i|$ and with the decrease of the relative lengths of the shells L/R this parameter increases, which leads in the final analysis to the appearance of scatter bands for the upper critical loads:

$$\varkappa_2 \leqslant \varkappa < \varkappa_1. \tag{11}$$

As long as the loading parameter is less than $\varkappa_2$, not a single shell out of the series being considered loses stability; when the loading parameter reaches the value $\varkappa_1$, all shells of the series being considered lose stability. Under the series of shells we understand those such that for the calculated number of degrees of freedom all absolute values of the coefficients $|f_i|$ of the trigonometric polynomial (7) coincide, the difference being connected only with the different signs. Practical calculations showed that the minimum loading parameter $\varkappa_2$ almost coincides with the loading parameter which is obtained in the case of such a dimensionless amplitude of faults:

$$f_0 = l|f_i|.$$

It is obvious that $f_0 \geq |f_*|$ [see (8)].

In Table 1 there is given a parameter $\varkappa_2 = 0.507$ ($|f_i| = 0.1$, $L/R = 0.5$) which is below the lower critical load for the problem being considered. The perturbation theory does not describe substantially nonlinear effects; the solution of the nonlinear system in the case of using methods of the perturbation theory [1-4] is expanded in a series with respect to a small parameter [see (1)]; the norm of the solution is not small when the upper critical load approaches the lower one. These nonlinear effects can be described, if we use in the solution of nonlinear problems the Bubnov–Galerkin method [12], especially selecting the approximation.

In Fig. 2 we have presented a scatter band of upper critical loads when the dimensionless amplitudes of the initial faults for shells with $R/t = 800$ and $L/R = 0.5$ vary; the curves 1 and 2 correspond to the minimum and maximum values of the upper critical loads, the curve 3 corresponds to the number of degrees of freedom in the calculation, while the curve 4 corresponds to the lower critical load. The curve 1 in the right lower corner of Fig. 2 is located below the line 4, since the theory of perturbations does not describe substantially nonlinear effects. The entire scatter band of "experimental" results presented in Fig. 2 is located below the curve for the upper critical load obtained in [4].

In Figs. 3 and 4 we have presented the results of calculations to determine the minimum critical loads (solid and dashed lines); in Fig. 4 we have given the dashed–dot curves for the number of degrees of freedom l. Figures 3 and 4 correspond to the values of the dimensionless amplitudes of the initial faults $|f_i| = 0.025$ and 0.05; curves 1-3 are constructed for shells with $L/R = 2$, 1, 0.5, with the solid curves for the parameter $b_2 = \min(-b)$ and the dashed curves for the parameter $b_1 = \max(-b)$. In Fig. 4, the curves 4 and 5 correspond to shells with $L/R = 0.5$ and 1. The calculated curves obtained for the critical loads are located considerably lower than the analogous curves from [4].

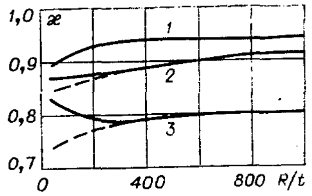

Fig. 3

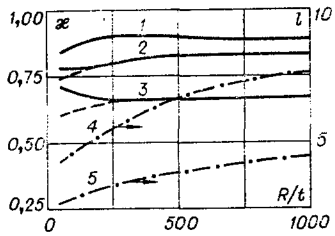

Fig. 4

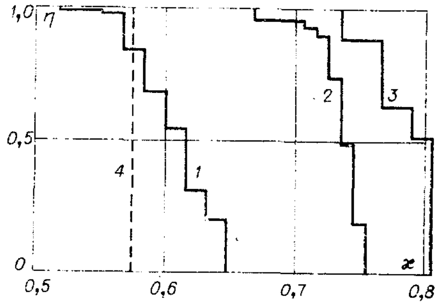

Fig. 5

The calculations carried out allow us to construct the probability characteristics of the process of stability loss [see (7), (11), and Fig. 2]; the derivation of the probability characteristics somewhat differs from the methods proposed, for example, in [13]. As a result of the numerical experiment, for each series we have obtained the set of critical loads

$$\varkappa_2 \leqslant \varkappa^1, \varkappa^2, \ldots, \varkappa^i \leqslant \varkappa_1, \tag{12}$$

where the parameter i is determined by the number of degrees of freedom l in the case being considered. Each event is assumed to be equally probable: In the trigonometric polynomial (7) we choose all possible combinations of the signs of the coefficients. Grouping in the corresponding manner the critical loads (12), we obtain the probability characteristics. In Fig. 5 we have presented the results of the probability calculations: The curve 1 corresponds to a shell with $R/t = 800$, $L/R = 0.5$, $|f_i| = 0.1$, $l = 9$; the curve 2 corresponds to $R/t = 600$, $L/R = 0.5$, $|f_i| = 0.05$, $l = 7$; the curve 3 corresponds to $R/t = 800$, $L/R = 1$, $|f_i| = 0.1$, $l = 5$. For a load parameter less than $\varkappa_2$ the probability for the shell not to lose stability is unity, while in the case when it exceeds $\varkappa_1$ the probability for the shell not to lose stability is zero. The scatter band in the examples presented is large; as a rule, in the neighborhood of the least parameter of the critical load $\varkappa_2$ there is an insignificant number of critical loads (12) for each series. The straight line 4, corresponding to the lower critical load, intersects the curve 1 (the perturbation theory does not take into account substantial nonlinearities).

It is desirable to take into account the characteristics of the stability process constructed from the numerical experiment when carrying out actual experiments. If the number of degrees of freedom l is large (which is to be borne in mind during the calculations), then a fairly substantial actual experiment is necessary in order to give with high confidence practical recommendations for the calculations.

LITERATURE CITED

1. P. Ya. Polubarinova-Kochina, "On the problem of stability of a plate," Prikl. Mat. Mekh., 3, No. 1 (1936).
2. W. T. Koiter, "Stability and postcritical behavior of elastic systems," Mekhanika, 63, No. 5 (1960).
3. J. Hutchinson and W. T. Koiter, "The theory of postcritical behavior of constructions," Mekhanika, 128, No. 4 (1971).
4. J. C. Amazigo and B. Budiansky, "Initial post-buckling behavior of cylindrical shells under external pressure," J. Math. Phys., 47, No. 3 (1968).
5. N. N. Bendich and V. M. Kornev, "On the density of eigenvalues in problems of stability of thin elastic shells," Prikl. Mat. Mekh., 35, No. 2 (1971).
6. V. M. Kornev, "On approximation in problems of stability and vibrations of elastic shells in the case of condensation of eigenvalues," Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, No. 2 (1972).
7. V. M. Kornev, "Peculiarities of problems of buckling of thin-walled shells," in: Dynamics of a Solid Body [in Russian], Izd. Inst. Gidrodinam. Sibirsk. Otd. Akad. Nauk SSSR, Novosibirsk (1976).
8. V. M. Kornev, "On the structural expressions in calculating cylindrical shells for vibrations and stability," Prikl. Mekh., 10, No. 4 (1974).
9. V. M. Kornev, "Stability of a circular cylindrical shell loaded by external transverse pressure with the boundary effect taken into account," Inzh. Zh. Mekh. Tverd. Tela, No. 3 (1967).
10. L. I. Shkutin and I. A. Shubin, "An experimental investigation into stability of shallow conical shells under static loading by pressure," Prikl. Mekh., 2, No. 6 (1966).
11. R. C. Tennyson, "Buckling modes of circular cylindrical shells under axial compression," AIAA J., 7, No. 8 (1969).
12. I. I. Vorovich, "On certain direct methods in the nonlinear theory of shallow shells," Prikl. Mat. Mekh., 20, No. 4 (1956).

13. V. V. Bolotin, Statistical Methods in Structural Mechanics [in Russian], 2nd ed., Stroiizdat, Moscow (1965).

STRESS CONCENTRATION NEAR THE APEX OF A CRACK BY THE COUPLE-STRESS THEORY OF ELASTICITY AND THE METHOD OF PHOTOELASTICITY

Ya. M. Shiryaev

UDC 539.30+620.171.5

The stress concentration near the apex of a crack in a transverse field of simple tension has already been the object of investigations within the framework of couple-stress elasticity theory [1-5]. It follows from [1-3] that the presence of couple-stresses results in a rise in the stress concentration near the crack apex. This result "contradicts" the reducing effect of couple-stresses in the known problem about the stress concentration near a circular hole in a tensile field. It is said in [4] that the presence of couple stresses does not influence the magnitude of the stress-intensity factor, and the stress concentration near an elliptical hole in a field of simple tension is considered in [5]. There results from an analysis of this paper that the stress concentration diminishes with the increase in a new elastic constant of the material l introduced by the couple-stress theory of elasticity.

Experimental papers in which the effect of the influence of couple-stresses on the stress concentration near a crack would be clarified are still nonexistent judging by the literature.

This paper is devoted to a clarification of the effect of the influence of couple stresses on stress concentration near a crack, both analytically and experimentally by the method of photoelasticity.

§1. The stress concentration near the apex of a crack in a transverse field of simple tension is considered in a coordinate system (Fig. 1).

It is useful to introduce the sum of normal stresses (invariant) into consideration, this sum having the same value in classical and couple-stress theory and being developed from the solution of the Dirichlet problem [6, 7], in order to determine the stress-intensity factor by means of couple-stress elasticity theory. It follows from the expression for the stresses [8] that

$$\sigma_x + \sigma_y = k(2/r)^{1/2}\cos(\theta/2), \tag{1.1}$$

where k is the stress-intensity factor.

Let us introduce the complex variable $z = x + iy = z_1 + re^{\theta i}$, where z_1 is a quantity characterizing the position of the crack apex. Then (1.1) can be written in the form

$$\sigma_x + \sigma_y = \mathrm{Re}\left[k\left(\frac{2}{z - z_1}\right)^{1/2}\right]. \tag{1.2}$$

Let us apply the complex variable method. According to [6, 7], the dependence between the left side of (1.2) and the stress functions has the form

$$\sigma_x + \sigma_y = 4\mathrm{Re}\,[\varphi'(z)]. \tag{1.3}$$

Comparing (1.2) and (1.3) and keeping in mind that (1.2) is valid only for values of z near z_1, we obtain the expression for k in the form

$$k = 2\sqrt{2}\lim_{z\to z_1}(z - z_1)^{1/2}\varphi'(z). \tag{1.4}$$

It is seen from (1.4) that the stress-intensity factors k are determined sufficiently simply if only the function $\varphi'(z)$ is known near the crack apex.

Leningrad. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 150-154, March-April, 1978. Original article submitted February 15, 1977.